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Fractional cointegration rank estimation

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Abstract

We consider cointegration rank estimation for a p-dimensional Fractional Vector Error Correction Model. We propose a new two-step procedure which allows testing for further long-run equilibrium relations with possibly different persistence levels. The first step consists in estimating the parameters of the model under the null hypothesis of the cointegration rank $r = 1, 2, \ldots, p-1$. This step provides consistent estimates of the order of fractional cointegration, the cointegration vectors, the speed of adjustment to the equilibrium parameters and the common trends. In the second step we carry out a sup-likelihood ratio test of no-cointegration on the estimated $p-r$ common trends that are not cointegrated under the null. The order of fractional cointegration is re-estimated in the second step to allow for new cointegration relationships with different memory. We augment the error correction model in the second step to adapt to the representation of the common trends estimated in the first step. The critical values of the proposed tests depend only on the number of common trends under the null, $p-r$, and on the interval of the orders of fractional cointegration b allowed in the estimation, but not on the order of fractional cointegration of already identified relationships. Hence this reduces the set of simulations required to approximate the critical values, making this procedure convenient for practical purposes. In a Monte Carlo study we analyze the finite sample properties of our procedure and compare with alternative methods. We finally apply these methods to study the term structure of interest rates.

Keywords: Error correction model, Gaussian VAR model, Likelihood ratio tests, Maximum likelihood estimation. **JEL:** C12, C15, C32.

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1 Introduction

Fractional cointegration generalizes standard models with $I(1)$ integrated time series and $I(0)$ cointegration relationships. In general, observed time series can display different orders of integration, while equilibrium relationships can be just characterized by a lower persistence or order of integration than the levels, perhaps allowing different values if there is more than one equilibrium relationship. Much focus of the literature has been placed on parameter estimation, using both semiparametric (e.g. Marinucci and Robinson (2001)) or parametric methods, which specify also short run dynamics (e.g. Robinson and Hualde (2003); Johansen and Nielsen (2012)). However, the estimation of the parameters of the cointegrated model assumes the knowledge of a positive number of cointegration relationships (and regression based methods also take the dependent variables as given), so the related testing problems on the existence of cointegration and the cointegration rank have also been investigated in the literature.

Fractional cointegration testing has been analyzed from different perspectives. One approach focuses on the estimation of the memory parameters, see e.g. Marinucci and Robinson (2001), Nielsen (2004), Gil-Alaña (2003), Robinson (2008). Marmol and Velasco (2004) and Hualde and Velasco (2008) compare OLS and different GLS-type estimates of the cointegrating vector to construct a test statistic. Łasak (2010) directly exploits a Fractional Vector Error Correction model (FVECM) to propose Likelihood Ratio (LR) tests for no-cointegration.

Recent work has proposed fractional cointegration tests inspired by multivariate methods. Breitung and Hassler (2002) solve a generalized eigenvalue problem of the type considered in the Johansen's procedure for developing multivariate score tests of fractional integration, see Johansen (1988, 1991, 1995) and Nielsen (2005). Avarucci and Velasco (2009) propose to exploit a parametric FVECM for the development of Wald tests of the cointegration rank. There have also been several semi-parametric proposals that focus on spectral matrix estimates, see Robinson and Yajima (2002), Chen and Hurvich (2003, 2006) and Nielsen and Shimotsu (2007).

We estimate the cointegration rank from a parametric perspective based on the specification of a FVECM. We rely on pseudo-LR tests based on restricted maximum likelihood (ML) estimates of the system. This is in contrast to Avarucci and Velasco (2009), who investigate the rank of unrestricted OLS estimates. We propose in this paper to perform a sequence of hypothesis tests based on a new two-stage method. It extends the results of testing the hypothesis of no-cointegration in Łasak (2010), of testing the cointegration rank in Johansen and Nielsen (2012), and of estimating the fractionally cointegration systems in Łasak (2008) and Johansen and Nielsen (2012). The first step of the proposed procedure consists in the estimation of the parameters of the FVECM under the null hypothesis of the cointegration rank $r = 1, 2, \ldots, p-1$. Under the null of the cointegration rank r, this estimation step provides consistent estimates of the order of fractional cointegration, the cointegration vectors and the speed of the adjustment to the equilibrium parameters, together with an approximation to the common trends. In the second step, we implement the no-cointegration sup LR tests considered in Łasak (2010) to the estimated common trends. The order of fractional cointegration is re-estimated in the second step, to allow for different persistence in the extra cointegration relationships. Our procedure results in tests statistics with asymptotic distribution depending only on the number of common trends under the null hypothesis of rank r, and on the interval of possible orders of cointegration, but not on the true order of cointegration, which can be seen as an advantage for an empirical work.

However, to adapt to the representation of the estimated common trends, we need to augment the error correction model in the second step to account for terms spanned by the cointegrating residuals. Then, parameter estimates are consistent and the cointegration test statistics of Łasak (2010) maintain the same asymptotic distribution as when original data is used, since parameter estimation from the first step is also shown to be asymptotically negligible. We analyze the performance of the proposed procedures in finite samples and compare our approach with the LR rank test of Johansen and Nielsen (2012). Their method imposes the assumption that all cointegration relationships share the same memory and results in an asymptotic distribution that depends on the true order of (fractional) cointegration. We also compare our tests with the benchmark LR test based on the standard VECM that assumes that the order of cointegration is known and equal to one, see Johansen (1988, 1991).

The reminder of the paper is organized as follows. Section 2 presents the basic FVECM, ML inference and sup-tests for no cointegration. Section 3 introduces our new two-step procedure for testing the cointegration rank. In Section 4 we present models with short run dynamics and discuss the generalization of our procedure for these models. Section 5 presents results of the Monte Carlo analysis. Section 6 contains the empirical analysis of the term structure of the interest rates. Section 7 concludes. The Appendix contains the proofs of our main results.

2 ML inference for fractional systems

In this section we introduce the basic FVECM, its ML estimation and ideas on cointegration testing that constitute the basis of our rank testing procedure presented in Section 3.

For a $p \times 1$ vector time series X_t, we consider the following representation

$$\Delta^d X_t = \Delta^{d-b} L_b \alpha \beta' X_t + \varepsilon_t, \tag{1}$$

where the fractional difference operator Δ^d is defined by the binomial expansion $\Delta^d = \sum_{j=0}^{\infty} (-1)^j \binom{d}{j} L^j$, L being the lag operator, d and b, respectively, orders of integration and cointegration satisfying $0 < b \leq d$, and $L_b = 1 - \Delta^b$, so that the filtered series $L_b X_t$ depends on lagged values of X_t but does not depend on the current value in period t. The coefficients α and β are $p \times r$ full rank matrices, $0 \leq r \leq p$, and ε_t is a $p \times 1$ vector of independent and identically distributed (i.i.d) errors with zero mean and positive definite variance-covariance matrix Ω. The matrix α contains the speed of adjustment to the equilibrium coefficients and β contains the cointegrating relationships. If $r = 0$, it implies that $\Pi = \alpha\beta' = 0$, so X_t is integrated of order d and no nontrivial linear combination of X_t has smaller order of integration. In the special case $r = p$, the matrix $\Pi = \alpha\beta'$ is unrestricted.

Equation (1) corresponds to a fractional $\mathrm{VAR}_{d,b}(0)$ model in Johansen and Nielsen (2012) and implies under some further conditions that there exists r, $0 < r < p$, different linear combinations β of the time series X_t that are integrated of order $d - b$, which is denoted by $I(d-b)$, while X_t is integrated of order d, i.e. $X_t \sim I(d)$. In Johansen and Nielsen (2012) the time series X_t is called a cofractional process of order $d-b$ with r, $r > 0$, being the cofractional or cointegration rank. Model (1) is encompassed by the fractional representations proposed in Granger (1986), Johansen (2008, 2009) and Avarucci and Velasco (2009) presented later in Section 4.

We assume that all initial values are set to zero, $X_t = \varepsilon_t = 0$, $t \leq 0$, so Δ^d can be replaced by Δ_+^d, i.e. the fractional filter truncated to positive values, $\Delta_+^d X_t = \Delta^d X_t 1\{t > 0\}$. The assumption that all initial values are zero is convenient to accommodate non square summable filters when $d \geq 0.5$. It is also possible to work conditional on a finite set of nonzero initial values for X_t but we prefer to keep the exposition as simple as possible.

Łasak (2010) has solved the problem of testing whether the system (1) is cointegrated searching for the true value of b in the interval $(0.5, d]$ and $d > 0.5$, so all potential cointegrating relationships are (asymptotically) stationary when $d < 1$ because then $d - b < 0.5$. The restriction $b > 0.5$ leads to asymptotics related to those of Johansen (1988) but based on fractional Brownian motions. ML estimation of the FVECM under the assumption that the cointegration rank r is known, $r > 0$, has been considered in Łasak (2008) and Johansen and Nielsen (2012) adapting Johansen's (1988) procedure. Johansen and Nielsen (2012) has derived the asymptotic distribution of the likelihood ratio test (LR) for testing any rank r, $0 \leq r < p$, which depends on the unknown order of fractional cointegration b. Note that when $r > 1$ all cointegrating relationships implied by the $\mathrm{VAR}_{d,b}(0)$ model have the same order of integration $d - b$. We do not maintain this restriction in our new rank testing procedure and we allow the extra cointegration relationships found in the second step to have a different order of integration within the interval $(0.5, d]$ than the relations found in the first step. It could be possible to develop a related procedure that searches for values of b smaller than

0.5, although the asymptotic theory would be different for these cases, see e.g. Avarucci and Velasco (2009).

We present the ML inference of the FVECM by reduced rank regressions for any $d > 0.5$. Define, omitting dependence on d, $Z_{0t} = \Delta^d X_t$ and $Z_{1t}(b) = \left(\Delta_+^{-b} - 1\right)\Delta^d X_t = \Delta_+^{d-b} L_b X_t$ and note that $Z_{1t}(b)$ does not depend on data at time t. Model (1) expressed in these variables becomes

$$Z_{0t} = \alpha\beta' Z_{1t}(b) + \varepsilon_t, \quad t = 1, \ldots, T.$$

Then, the log-likelihood function, log $\mathcal{L}_r$, for the model (1), under the hypothesis of r cointegrating relationships and the gaussianity of ε_t, is given, apart from a constant, by

$$\log \mathcal{L}_r(\alpha, \beta, \Omega, b) = -\frac{T}{2}\log|\Omega| - \frac{1}{2}\sum_{t=1}^{T}[Z_{0t} - \alpha\beta' Z_{1t}(b)]'\Omega^{-1}[Z_{0t} - \alpha\beta' Z_{1t}(b)].$$

For fixed b the maximum of the likelihood is obtained by solving the eigenvalue problem

$$\left|\lambda_i(b)S_{11}(b) - S_{10}(b)S_{00}^{-1}S_{01}(b)\right| = 0 \tag{2}$$

for eigenvalues $\lambda_i(b)$ (ordered by decreasing magnitude for $i = 1, \ldots, p$) and sample cross moments

$$S_{jk}(b) = T^{-1}\sum_{t=1}^{T} Z_{jt}(b)Z_{kt}(b)' \quad j, k = 0, 1,$$

where S_{jk} is a function of b except when $j = k = 0$. The parameter b is estimated by maximizing the concentrated likelihood in a compact set $\mathcal{B} \subset (0.5, d]$, i.e.

$$\hat{b}_r = \arg\max_{b\in\mathcal{B}} \mathcal{L}_r(b),$$

where we can write

$$\mathcal{L}_r(b) = \left[|S_{00}| \prod_{i=1}^{r}(1 - \lambda_i(b))\right]^{-T/2} \tag{3}$$

when estimation is done under the hypothesis

$$H_r : rank\ (\Pi) = r.$$

Expression (3) can be used to construct the sequence of LR tests for testing the fractional cointegration rank in the model (1). The first step is to test the null of no cointegration, $H_0 : rank\ (\Pi) = 0$. We can test it against two different alternatives, full cointegration rank of the impact matrix $\Pi = \alpha\beta'$, i.e. $H_p : rank\ (\Pi) = p$, or one extra cointegrating relationship,

$H_1 : rank\ (\Pi) = 1$.

Łasak (2010) has described how to test H_0 against H_p and H_1. The LR statistic for testing H_0 against H_p (sup trace test) is defined by

$$\mathcal{LR}_T^p\,(0|p) = -2\log\left[\mathcal{L}_0/\mathcal{L}_p\left(\hat{b}_p\right)\right] = -T\sum_{i=1}^{p}\log[1-\lambda_i(\hat{b}_p)], \tag{4}$$

where $\hat{b}_p = \arg\max_{b\in\mathcal{B}}\mathcal{L}_p(b)$, $\mathcal{L}_p$ is the likelihood under the hypothesis H_p of rank p and $\mathcal{L}_0 = |S_{00}|^{-T/2}$ is the likelihood when $r = 0$.

Alternatively, the LR statistic for testing H_0 against H_1 (sup maximum eigenvalue test) is defined by

$$\mathcal{LR}_T^p\,(0|1) = -2\log\left[\mathcal{L}_0/\mathcal{L}_1\left(\hat{b}_1\right)\right] = -T\log[1-\lambda_1(\hat{b}_1)], \tag{5}$$

where $\hat{b}_1 = \arg\max_{b\in\mathcal{B}}\mathcal{L}_1(b)$ and $\mathcal{L}_1$ denotes the likelihood under the hypothesis of rank 1, H_1. Recall that under the null of no cointegration ($r = 0$) we cannot hope that $\hat{b}_1$ or $\hat{b}_p$ estimate consistently a nonexisting true value of b in model (1), and because of that the LR tests (4) and (5) can be interpreted as sup LR tests, in the spirit of Davies (1977) and Hansen (1996).

Łasak (2010) has investigated the asymptotic distributions of the test statistics (4) and (5) under H_0 and Assumption 1.

Assumption 1 *ε_t are i.i.d. vectors with mean zero, positive definite covariance matrix Ω, and $E||\varepsilon_t||^q < \infty,\ q \geq 4,\ q > 2/\left(2\underline{b}-1\right),\ \underline{b} = \min\mathcal{B} > 0.5$, where $\mathcal{B} \subset (0.5, d]$ is a compact set.*

Then, under the null hypothesis of no cointegration H_0,

$$\mathcal{LR}_T^p\,(0|p) \xrightarrow{d} \sup_{b\in\mathcal{B}} trace\left[\pounds_p(b)\right] \ \stackrel{def}{=}\ J_p \tag{6}$$

and

$$\mathcal{LR}_T^p\,(0|1) \xrightarrow{d} \sup_{b\in\mathcal{B}} \lambda_{\max}\left[\pounds_p(b)\right] \ \stackrel{def}{=}\ E_p, \tag{7}$$

where

$$\pounds_p(b) = \int_0^1 (dB)\,B_b'\left[\int_0^1 B_bB_b'du\right]^{-1}\int_0^1 B_b\,(dB)', \tag{8}$$

B_b is a p-dimensional standard fractional Brownian motion with parameter $b \in \mathcal{B}$, $B_b\,(x) = \Gamma^{-1}\,(b)\int_0^x (x-z)^{b-1}\,dB\,(z)$, $B = B_1$ is a standard Brownian motion on the unit interval and Γ is the Gamma function. Łasak (2010) has obtained by simulation the quantiles of the asymptotic distributions in (6) and (7) for the interval $\mathcal{B} = [0.5;\ d]$, when $d = 1$. In this case, the restrictions $d = 1$ and $b > 0.5$ imply that the test focuses on deviations from equilibrium that are asymptotically stationary of any magnitude.

When we reject the null hypothesis H_0 of no cointegration we only obtain the information that the system (1) is cointegrated, but we do not know how many cointegration relationships share the elements of X_t, so we need to proceed further and solve the problem of the cointegration rank estimation. For testing the cointegration rank r against rank p, $r = 1, \ldots, p-1$ in model (1) we can use the general LR tests proposed by Johansen and Nielsen (2012) based on the solutions of the eigenvalue problem (2) under both hypothesis, i.e.

$$\mathcal{LR}_T^p\left(r|p\right) = -2T\log\left[\mathcal{L}_r\left(\hat{b}_r\right)/\mathcal{L}_p\left(\hat{b}_p\right)\right] = -T\left\{\sum_{i=1}^{p}\log[1-\lambda_i(\hat{b}_p)] - \sum_{i=1}^{r}\log[1-\lambda_i(\hat{b}_r)]\right\}, \tag{9}$$

where estimates of the cointegration order under the null $(\hat{b}_r)$ and under the alternative $(\hat{b}_p)$ are different in general. The null asymptotic distribution of the test statistic $\mathcal{LR}_T\left(r|p\right)$ for $b_0 > 0.5$, trace$\{\pounds_{p-r}(b_0)\}$, depends on the true cointegration order, while is $\chi^2((p-r)^2)$ when $b_0 < 0.5$. Johansen and Nielsen (2012) suggest using the computer program by MacKinnon and Nielsen (2013) to obtain critical values for the tests when $b_0 > 0.5$.

In the next section we propose a new two-step procedure that leads to tests with the same null asymptotic distributions as tests (4) and (5), which do not depend on any nuisance parameters other than the number of the common trends under the null, $p-r$, and the interval $\mathcal{B}$ which can be fixed arbitrarily close to $(0.5, d]$.

3 New tests for the cointegration rank

In this section we propose a new two-step procedure to establish the cointegration rank in the FVECM given in (1). This procedure extends the idea of testing the null of no cointegration in Łasak (2010) and testing the cointegration rank in Johansen and Nielsen (2012). The main novelty of our proposal is that different cointegration relations are allowed to have different persistence. It leads to null asymptotic distributions based on (8) as for cointegration testing.

Our method exploits Granger's representation for the cofractional VAR model. From Theorem 2 in Johansen and Nielsen (2012), we can represent the cointegrated system (1) as

$$X_t = C\Delta_+^{-d}\varepsilon_t + \Delta_+^{b-d}Y_t^+,$$

where $C = \beta_\perp\left(\alpha'_\perp\beta_\perp\right)^{-1}\alpha'_\perp$ and Y_t^+ is fractional of order zero, with initial conditions set to zero and $\det\left(\alpha'_\perp\beta_\perp\right) \neq 0$. Then, when projecting X_t in the direction $\beta_\perp$,

$$\beta'_\perp X_t = \beta'_\perp C\Delta_+^{-d}\varepsilon_t + \Delta_+^{b-d}\beta'_\perp Y_t^+, \tag{10}$$

where $\beta'_{\perp}C$ is of rank $p-r$ under the null H_r, so that the $p-r$ series $\beta'_{\perp}X_t$ are just a rotation of the $I(d)$ common trends $\alpha'_{\perp}\Delta_{+}^{-d}\varepsilon_t$ plus the $I(d-b)$ components $\Delta_{+}^{b-d}\beta'_{\perp}Y_t^{+}$. Therefore, under H_r, $\beta'_{\perp}X_t$ is a non cointegrated $(p-r)\times 1$ vector of $I(d)$ series.

By contrast, under an alternative H_{r+r_1} generated by the model

$$\Delta^d X_t = \left(\alpha\beta' + \alpha_1\beta_1'\right)\left(\Delta_{+}^{-b} - 1\right)\Delta^d X_t + \varepsilon_t, \tag{11}$$

where the $p\times r$ matrices α and β are of rank r, and the $p\times r_1$ matrices α_1 and β_1 are of rank r_1, $p-r \geq r_1 > 0$, estimation under the null H_r cannot account for all the existing $r+r_1$ cointegrating relationships. That is, any $p\times r$ vector β can only capture at most r out of the $r+r_1$ cointegrating directions so that $\beta'_{\perp}X_t$ must contain at least one further cointegration relationship, and this should be detected by any fractional cointegration test such as Łasak's (2010).

These intuitions lead to a two step testing procedure. The first step consists in ML estimation of model (1) under the null hypothesis H_r of cointegration rank r. This provides consistent estimates of b and of the decomposition $\Pi = \alpha\beta'$, where α and β are $p\times r$ matrices, as in Theorem 10 of Johansen and Nielsen (2012). Then we compute (super) consistent estimates $\hat{\beta}_{\perp}$ of the full rank $p\times(p-r)$ matrix $\beta_{\perp}$ satisfying $\beta'_{\perp}\beta = 0$ and the proxies of the $p-r$ common trends $\hat{\beta}'_{\perp}X_t$.

The second step of our testing procedure exploits the fact that under the null H_r the estimated common trends $\hat{\beta}'_{\perp}X_t$ are not cointegrated, but must be cointegrated under the alternative. Then, to test for the presence of additional cointegrating relationships in $\hat{\beta}'_{\perp}X_t$, we propose to implement the sup LR tests (4) and (5) of the null of no cointegration described in Section 2 to the $p-r$ series $\hat{\beta}'_{\perp}X_t$ using critical values from the J_{p-r} and E_{p-r} distributions (see (6) and (7)). Given the consistency of $\hat{\beta}$ and therefore of $\hat{\beta}_{\perp}$, replacing $\beta_{\perp}$ by $\hat{\beta}_{\perp}$ in $\hat{\beta}'_{\perp}X_t$ does not affect the asymptotic null distribution of the tests if we further augment the model to accommodate the extra $I(d-b)$ term in (10) that is not present in model (1) when $\Pi = \alpha\beta' = 0$.

This approach has two particular characteristics. First, when searching for further cointegration relationships among the estimated common trends, it does not restrict b to the first-step estimate $\hat{b}_r$ of the persistence of the cointegrating relationships under the null. Second, the linear combinations $\beta'_{\perp}X_t$ are not pure $I(d)$ processes, as it is implied by (1) for the original series X_t when rank$(\Pi) = 0$. Our testing regressions take into account this particular feature of the projections $\beta'_{\perp}X_t$ compared to the data generated under (1) by introducing an augmentation term. This augmentation is derived for the case of triangular systems, which are easier to handle as we show next.

Consider the triangular representation of a fractionally cointegrated $I(d)$ vector with rank r,

$$\begin{aligned}\beta' X_t &= \Delta_+^{b-d} u_{1t}, \\ \gamma' X_t &= \Delta_+^{-d} u_{2t},\end{aligned} \tag{12}$$

see Johansen (2008, pp. 652-53), where β and γ are, respectively, $p \times r$ and $p \times (p-r)$ matrices, $u_t = (u'_{1t}, u'_{2t})'$ is $iid\,(0, \Sigma)$, with $\Sigma > 0$, and $\Theta = (\beta \vdots \gamma)$ has full rank p. Then we can write

$$\begin{aligned}\beta' \Delta^d X_t &= \left(\Delta - \Delta_+^{d-b}\right) \beta' X_t + u_{1t}, \\ \gamma' \Delta^d X_t &= u_{2t},\end{aligned}$$

so that from the identity $\gamma_\perp \left(\beta' \gamma_\perp\right)^{-1} \beta' + \beta_\perp \left(\gamma' \beta_\perp\right)^{-1} \gamma' = I_p$, it follows that the system admits the FVECM (1) with $\alpha = -\gamma_\perp \left(\beta' \gamma_\perp\right)^{-1}$ and $\varepsilon_t = K u_t$ where $K = (\gamma_\perp \left(\beta' \gamma_\perp\right)^{-1} \vdots \beta_\perp \left(\gamma' \beta_\perp\right)^{-1})$. Therefore we obtain the representation

$$X_t = \mathbf{\Theta}^{-1\prime} \left(\begin{array}{c} \Delta_+^{b-d} u_{1t} \\ \Delta_+^{-d} u_{2t} \end{array} \right),$$

and hence

$$\beta'_\perp X_t = M_1 \Delta_+^{b-d} u_{1t} + M_2 \Delta_+^{-d} u_{2t} \tag{13}$$

where M_2 is a $(p-r) \times (p-r)$ full rank matrix so that there is no β_1 such that $\beta'_1 \left(\beta'_\perp X_t\right)$ is an $I\,(d - b_1)$ process, for any $b_1 > 0$, i.e. a process less integrated than $\beta'_\perp X_t$. However, as far as $M_1 \neq 0$, $\beta'_\perp X_t$ contains some $I\,(d-b)$ terms, by contrast with equation (1) when $r = 0$ and $\Pi = 0$. The interesting feature of the triangular model is that these $I\,(d-b)$ terms are spanned by the cointegrating residuals $\beta' X_t = \Delta_+^{b-d} u_{1t}$.

Then, noting that from (13),

$$\beta'_\perp \Delta^d X_t = M_1 \left(\Delta_+^b - 1\right) u_{1t} + M_1 u_{1t} + M_2 u_{2t}, \tag{14}$$

a reduced rank regression of $\hat{V}_{0t} = \hat{\beta}'_\perp \Delta^d X_t$ on $\hat{V}_{1t}\,(b_1) = (1 - \Delta_+^{-b_1}) \hat{\beta}'_\perp \Delta^d X_t$ has to control for the predictable term $M_1 \left(\Delta_+^b - 1\right) u_{1t}$ in the right hand side of (14) to estimate consistently the true coefficient $\Pi_1 = 0$ under H_r. As a proxy for u_{1t} we use the linear projection of $\hat{\beta}' \Delta^d X_t$ given $\varepsilon_t \left(\hat{b}, \hat{\alpha}, \hat{\beta}\right)$,

$$\tilde{u}_{1t} = \left(\sum_{t=1}^T \hat{\beta}' \Delta^d X_t \varepsilon_t \left(\hat{b}, \hat{\alpha}, \hat{\beta}\right)' \right) \left(\sum_{t=1}^T \varepsilon_t \left(\hat{b}, \hat{\alpha}, \hat{\beta}\right) \varepsilon_t \left(\hat{b}, \hat{\alpha}, \hat{\beta}\right)' \right)^{-1} \varepsilon_t \left(\hat{b}, \hat{\alpha}, \hat{\beta}\right), \tag{15}$$

which identifies the contemporaneous contribution of u_{1t} in $\beta'\Delta^d X_t = \Delta_+^b u_{1t}$ out of the first-step residuals $\varepsilon_t\left(\hat{b},\hat{\alpha},\hat{\beta}\right)$ under H_r, $\varepsilon_t(b,\alpha,\beta) = \left(I_p - \alpha\beta'(\Delta_+^{-b}-1)\right)\Delta^d X_t$. Then we augment the FVECM of $\hat{V}_{0t}$ with the filtered series $\left(\Delta_+^{\hat{b}}-1\right)\tilde{u}_{1t}$,

$$\hat{V}_{0t} = \Pi_1 \hat{V}_{1t}(b_1) + \Phi\left(\Delta_+^{\hat{b}}-1\right)\tilde{u}_{1t} + \text{error}_t, \tag{16}$$

and fit the model by reduced rank regression.

Then our two-step rank testing procedure is as follows:

Step 1. Estimate the model (1) under the null H_r for the original data $\Delta^d X_t$ and recover the common trends increments $\hat{V}_{0t} = \hat{\beta}_\perp'\Delta^d X_t$, the cointegrating residuals increments $\hat{\beta}'\Delta^d X_t$ and the model residuals $\varepsilon_t\left(\hat{b},\hat{\alpha},\hat{\beta}\right)$.

Step 2. Compute the LR statistics for testing $\text{rank}(\Pi_1)=0$ against $\text{rank}(\Pi_1)=p-r$ and $\text{rank}(\Pi_1)=1$, denoted as $\mathcal{LR}_T^{p-r}(0|p-r)$ and $\mathcal{LR}_T^{p-r}(0|1)$, see (4) and (5), respectively, from the augmented FVECM for $\hat{V}_{0t}$ given in regression (16).

We next show that, paralleling cointegration testing, the null asymptotic distributions of these LR test statistics are E_{p-r} and J_{p-r}, respectively, since replacing $\beta_\perp$ by $\hat{\beta}_\perp$ and b_0 by $\hat{b}$ has no asymptotic impact on the test statistics under Assumption 2.

Assumption 2

$$\hat{\beta}-\beta = O_p\left(T^{-1/2}\right),\ \hat{\alpha}-\alpha = O_p\left(T^{-1/2}\right)\ \textit{and}\ \hat{b}-b_0 = O_p\left(T^{-1/2}\right).$$

Then we present our first result, whose proof is contained in the Appendix, as well as other proofs.

Theorem 1 *Under Assumptions 1, 2 and model* (12), *the LR tests based on regression (16) for testing rank*$(\Pi_1)=0$, *satisfy under the null hypothesis* H_r,

$$\mathcal{LR}_T^{p-r}(0|1) \xrightarrow{d} E_{p-r},$$
$$\mathcal{LR}_T^{p-r}(0|p-r) \xrightarrow{d} J_{p-r}.$$

The consistency rate for the ML estimates of $\hat{\beta}$ is T^{b_0} for $0.5 < b_0 \leq d$ and $T^{1/2}$ for $b_0 < \min\{0.5, d\}$ from Theorems 6 and 10 in Johansen and Nielsen (2012), so with Assumption 2 we are not imposing a lower bound on the true value of b, i.e. on the strength of the cointegrating relationships under the null. However, the null asymptotic distribution in Theorem 1 requires that the set $\mathcal{B}$ only contains values of b_1 larger than 0.5, given that

$d > 0.5$. Therefore, only the degrees of freedom of E_{p-r} and J_{p-r} need to be adapted for the dimension of $\hat{\beta}'_{\perp} X_t$ under H_r, i.e. $p-r$, compared to the cointegration test for the null H_0 as in the usual unit root framework. These distributions do not depend on any further nuisance parameter other than the set $\mathcal{B}$, which can be taken as $[0.5+\epsilon, d]$ for $\epsilon > 0$ arbitrarily small.

For the analysis of the consistency of our tests we can consider the alternative hypothesis H_{r+r_1} generated by the model (11). Since $\hat{\beta}_{\perp}$ is of dimension larger than the null space of the actual cointegrating matrix $(\beta \; \beta_1)$ under H_{r+r_1}, $\hat{\beta}'_{\perp} X_t$ still contains at least one further cointegration relationship. Then, the consistency of the test would follow from the correlation between $\hat{\beta}'_{\perp} \Delta^d X_t$ and $\left(\Delta_+^{-b_1} - 1\right) \hat{\beta}'_{\perp} \Delta^d X_t$ under H_{r+r_1} for a range of values of b_1 and any full rank $p \times (p-r)$ matrix $\hat{\beta}_{\perp}$ as in the usual test for cointegration.

If the value of the parameter d is unknown and has to be estimated, then we replace $\hat{V}_{0t}$ and $\hat{V}_{1t}(b_1)$ by $\hat{V}_{0t}\left(\hat{d}\right) = \hat{\beta}'_{\perp} \Delta^{\hat{d}} X_t$ and $\hat{V}_{1t}\left(b_1, \hat{d}\right) = (1 - \Delta_+^{-b_1}) \hat{\beta}'_{\perp} \Delta^{\hat{d}} X_t$ in the test statistics and possibly readjust the set $\mathcal{B}$. Then the following corollary justifies this policy, being similar to Theorem 1 in Robinson and Hualde (2003).

Corollary 2 *The conclusions of Theorem 1 remain valid if* $\Delta^d X_t$ *is replaced by* $\Delta^{\hat{d}} X_t$ *and* $\hat{d} - d = O_p\left(T^{-1/2}\right)$.

It is also possible to consider situations where elements of X_t have different memory so that model (1) is generalized as

$$\mathbf{\Delta^d} X_t = \Delta^{-b} L_b \alpha \beta' \mathbf{\Delta^d} X_t + \varepsilon_t$$

where

$$\mathbf{\Delta^d} X_t = \begin{pmatrix} \Delta^{d_1} X_t \\ \vdots \\ \Delta^{d_p} X_t \end{pmatrix},$$

$d_1 = d_2 \geq d_3 \geq \cdots \geq d_p$. Then we can proceed using our procedure as usual just replacing vector $\Delta^{\hat{d}} X_t$ by the vector $\Delta^{\hat{\mathbf{d}}} X_t = \left(\Delta^{\hat{d}_1} X_t, \ldots, \Delta^{\hat{d}_p} X_t\right)'$ and with a similar interpretation of memory reduction of the magnitude b for linear combinations of $\left(\Delta^{d_1} X_t, \ldots, \Delta^{d_p} X_t\right)'$ in the direction β. Further, our method is also valid for series that have a nonzero mean μ, i.e. when observed data is given by $\mu + X_t$, since these series also satisfy equation (1) because $\Delta^d 1 = \Delta^{d-b} 1 = 0$ when $d - b > 0$, as noted by Johansen and Nielsen (2012).

4 Rank testing in FVECM with short run dynamics

To make the FVECM (1) more flexible, a natural idea is to add a lag structure in terms of fractional lags of $\Delta_+^d X_t$ to produce a $\text{VAR}_{d,b}(k)$ model,

$$\Delta_+^d X_t = \Delta_+^{d-b} L_b \alpha\beta' X_t + \sum_{i=1}^{k} \Gamma_i L_b^i \Delta_+^d X_t + \varepsilon_t, \tag{17}$$

as in Johansen (2008, 2009). In this case $\Delta_+^d X_t$ follows a VAR model in the lag operator $L_b = \left(1 - \Delta_+^b\right)$ rather than in the usual lag operator $L = L_1$. Johansen and Nielsen (2012) show that the existence of a Granger representation for X_t depends on $\det(\alpha'_\perp \Gamma \beta_\perp) \neq 0$ with $\Gamma = I_p + \sum_{i=1}^{k} \Gamma_i$, $\Gamma_k \neq 0$, and on the roots of the matrix polynomial $\Psi(y) = (1-y) I_p - \alpha\beta' y - \sum_{i=1}^{k} \Gamma_i (1-y) y^i$.

The representations for the common trends from model (17) are not amenable for developing our two-step procedure because lags depend on b, but following Avarucci and Velasco (2009) we allow for short run correlation in the levels of X_t using ordinary lags by assuming that the prewhitened series $X_t^\dagger = A(L) X_t$ satisfy the model (1), but we actually observe X_t, i.e.

$$\Delta_+^d X_t = \Delta_+^{d-b} L_b \alpha\beta' A(L) X_t + (I - A(L)) \Delta_+^d X_t + \varepsilon_t, \tag{18}$$

where $A(L) = I - A_1 L - \cdots - A_k L^k$. This model can be shown to encompass triangular models used in the literature (cf. Robinson and Hualde (2003)) and has also nice representations if the roots of the equation $det[A(z)] = 0$ are out of the unit circle, $d > b$. In fact, if $X_t^\dagger$ is cointegrated with cointegrating vector β, X_t is also cointegrated with cointegrating vector in the same space spanned by β given that $A(1)$ is full rank.

Even under the assumption of known d, model (18) is nonlinear in $\Pi = \alpha\beta'$ and $A_1, \ldots, A_k$, so ML estimation can not be performed through the usual procedure of prewhitening the differenced levels $Z_{0t} = \Delta^d X_t$ and the fractional regressor $Z_{1t}(b) = \Delta_+^{d-b} L_b X_t$ given particular values of d and b. However, it is easier to estimate the unrestricted linear model (in A_j and A_j^*) given by

$$Z_{0t} = \alpha\beta^{*\prime} Z_{1t}(b) + \sum_{j=1}^{k} A_j^* \Delta Z_{1t+1-j}(b) + \sum_{j=1}^{k} A_j Z_{0t-j} + \varepsilon_t, \tag{19}$$

under the assumption of α and β^* being $p \times r$, without imposing $A_j^* = -\Pi \tilde{A}_j$. In (19) $\beta^* = A(1)' \beta$ spans the same cointegration space as β and we have used the decomposition $A(L) = A(1) - \Delta \tilde{A}(L)$ so that the coefficients of $\tilde{A}(L) = \sum_{j=0}^{k-1} \tilde{A}_j L^j$ satisfy $\tilde{A}_j = \sum_{i=1+j}^{k} A_i$, $j = 0, \ldots, k-1$. The estimation procedure follows as in the usual reduced rank regression but with an initial step to prewhiten the series Z_{0t} and $Z_{1t}(b)$ on k lags of Z_{0t} and $\Delta Z_{1t+1}(b)$. This estimation could be inefficient compared to ML, but is much simpler to compute and analyze.

To test for the cointegration rank, we can construct the linear combinations $\hat{V}_{0t} = \hat{\beta}'_{\perp}\Delta^d X_t$ and $\hat{V}_{1t}(b_1) = (1-\Delta_+^{-b_1})\hat{\beta}'_{\perp}\Delta^d X_t$ given the first-step estimates of β and b under the null H_r, $r>0$, and propose a similar second-step testing regression equation as for $k=0$. In this case the FVECM has to be enlarged by proxies of $\left(\Delta_+^b - 1\right) u_{1t}$ as well as by lags of $\Delta^d X_t$,

$$\hat{V}_{0t} = \Pi_1 \hat{V}_{1t}(b_1) + \sum_{j=1}^{k} C_j \Delta^d X_{t-j} + \Phi\left(\Delta_+^{\hat{b}} - 1\right)\tilde{u}_{1t} + \text{error}_t. \tag{20}$$

As when $k=0$, $\tilde{u}_{1t}$ is obtained as in (15) from a projection of $\hat{\beta}'\Delta^d X_t$ on the FVECM residuals $\varepsilon_t\left(\hat{b},\hat{\alpha},\hat{\beta},\hat{A}^*,\hat{A}\right)$ from (19) to isolate the u_{1t} contribution in $\beta'\Delta^d X_t$, which might contain other predictable contributions at time t due to the autoregressive structure. This can be seen in a triangular model set up with the VAR modelization $A(L)X_t = X_t^\dagger$ in levels and $X_t^\dagger$ generated by (12) so that

$$X_t = (I - A(L))X_t + \mathbf{\Theta}^{-1\prime}\begin{pmatrix} \Delta_+^{b-d}u_{1t} \\ \Delta_+^{-d}u_{2t} \end{pmatrix}, \tag{21}$$

and therefore

$$\beta'_{\perp}X_t = \sum_{j=1}^{k}\beta'_{\perp}A_j X_{t-j} + M_1\Delta_+^{b-d}u_{1t} + M_2\Delta_+^{-d}u_{2t},$$

with M_2 being full rank under H_r, justifying regression (20).

In sum, our two-step testing procedure in the presence of short run dynamics is as follows:

Step 1. Estimate the model (19) under the null H_r for the original data $Z_{0t} = \Delta^d X_t$ with the augmentation terms $(\Delta Z_{1t+1-j}(b), Z_{0t-j})$, $j=1,\ldots,k$, and recover the common trends increments $\hat{\beta}'_{\perp}\Delta^d X$ and the model residuals $\varepsilon_t\left(\hat{b},\hat{\alpha},\hat{\beta},\hat{A}^*,\hat{A}\right)$.

Step 2. Compute the LR statistics for testing rank$(\Pi_1)=0$ against rank$(\Pi_1)=p-r$ and rank$(\Pi_1)=1$, $\mathcal{LR}_T^{p-r}(0|p-r)$ and $\mathcal{LR}_T^{p-r}(0|1)$, see (4) and (5), respectively, from the augmented FVECM (20) for $\hat{V}_{0t} = \hat{\beta}'_{\perp}\Delta^d X_t$.

Theorem 3 shows that the asymptotic null distributions of the trace and maximum eigenvalue cointegration test statistics based on (20) remain J_{p-r} and E_{p-r}, respectively, if the first step estimates converge fast enough.

Theorem 3 *Under Assumptions 1, 2, model (21) and*

$$\hat{A}_j^* - A_j^* = O_p\left(T^{-1/2}\right) \quad \textit{and} \quad \hat{A}_j - A_j = O_p\left(T^{-1/2}\right),\ j = 1, \ldots, k,$$

the LR tests for testing $\Pi_1 = 0$ *based on regression* (20) *have the same asymptotic distribution under the null* H_r *as in Theorem 1.*

5 Finite sample properties of cointegration rank tests

In this section we analyze the performance of the proposed new procedure in finite samples. We simulate a cointegrated trivariate system ($p = 3$), with $d = 1$, using the following triangular representation

$$X_t = \begin{pmatrix} I_r & \delta \\ 0 & I_{p-r} \end{pmatrix} \begin{pmatrix} \Delta_+^{b-1} u_{0t} \\ \Delta_+^{b_1-1} u_{1t} \\ \Delta_+^{-1} u_{2t} \end{pmatrix},\ t = 1, \ldots, T, \tag{22}$$

which implies the FVECM (1) with

$$\alpha = \begin{pmatrix} -I_r \\ 0 \end{pmatrix} \text{ and } \beta' = \left(I_r \ \ -\delta\right).$$

The innovations $u_t = (u'_{0t}, u'_{1t}, u'_{2t})'$ are independent standard Gaussian *iid*.

To investigate the empirical size of the tests we simulate (22) with cointegration rank $r = 1$ and cointegrating vector $\beta = [1\ 0\ -1]'$ and for the power study, when $r = 2$, we add an extra cointegrating relationship $\beta_1 = [0\ 1\ -0.5]'$. Further we also consider the model with short run dynamics (18) and with $k = 1$. For this model we add to (22) the autoregression

$$Y_t = A_1 Y_{t-1} + X_t,$$

with $Y_0 = 0$ and $A_1 = a\ I_p$, where $a = 0.5$ or $a = 0.8$.

We simulate the systems with the memory of the first cointegrating relationship determined by $b = 0.4, 0.51, 0.6, 0.7, 0.8, 0.9, 0.99$, which covers the cases of strong ($b > 0.5$) and weak cointegration ($b = 0.4$) of the existing cointegration relationship under $r = 1$, but for our two step tests we always set $\mathcal{B} = [0.5, 1]$, which is only determined by the value $d = 1$. For the power analysis the memory of the second cointegrating relationship is $b_1 = b, 0.20, 0.51, 0.9$. This way we can illustrate the power of the testing procedure when the memory $d - b$ of the second cointegrating relationship is the same as the memory of the first cointegrating relationship and when is relatively large or small, including the case $b_1 = 0.20$ which is smaller than the lower bound of $\mathcal{B} = [0.5, 1]$ and all b's. The sample size is set to $T = 50, 100, 200, 400$ for size simulations and $T = 50,\ 100$ for power analysis. For all simulations we use OxMetrics 7.00, see Doornik and Ooms (2007) and Doornik (2009 a,b) and we perform $10,000$ repetitions of each experiment.

We compare the performance of the following tests discussed in this paper, i.e.:

1. New two step procedures, i.e. trace test, 2s-$\mathcal{LR}_T^2\,(0|2)$, and maximum eigenvalue test, 2s-$\mathcal{LR}_T^2\,(0|1)$, based on the FVECM for $\hat{\beta}_{\perp}' X_t$, with the additional control $\left(\Delta^{\hat{b}} - 1\right) \tilde{u}_{1t}$ as in (16).
2. Trace and maximum eigenvalue LR tests, $\mathcal{LR}_T^3\,(1|3)$ and $\mathcal{LR}_T^3\,(1|2)$ respectively, based on the standard VECM with $d = b = 1$ like in Johansen (1988, 1991), called Johansen's trace and Johansen's maximum eigenvalue tests.
3. Trace LR test $\mathcal{LR}_T^3\,(1|3)$ proposed by Johansen and Nielsen (2012), where estimation is restricted to $d = 1$ and critical values are obtained from the computer program of MacKinnon and Nielsen (2013) with ML estimate of b rounded to a decimal point.

The asymptotic distribution of Johansen's tests in 2. is not justified for the data generating process (22), as they are based on a misspecified model. However we check their performance, since they are included in most econometric packages and they are routinely used by practitioners. Similarly, Johansen and Nielsen (2012) test in 3. is only correctly specified when $k = 0$, but not when $k = 1$, since it uses model (17) with fractional lags L_b instead of (18) which is used to simulate data.

The results of our size simulations are presented in Tables 1-3. Table 1 provides the percentage of rejections under the null hypothesis of cointegration rank $r = 1$ for $k = 0$ and Tables 2 and 3 for $k = 1$ and for $a = 0.5$ and $a = 0.8$, respectively. When $k = 0$ the new two step procedures are undersized for all sample sizes considered but improve slowly for larger samples. For moderate and large sample sizes, rejections do not change much with b, including $b = 0.4$. The trace LR test by Johansen and Nielsen (2012) is usually oversized, but size distortions are decreasing with sample size T and true value b. Johansen's LR tests have size close to the nominal 5% in all considered cases, except of $b = 0.4$, for moderate T, see Table 1. When $k = 1$ the two step procedures have higher empirical size than when $k = 0$, being slightly oversized in smaller samples, but simulated size tends to decrease with T. When $k = 1$ Johansen's tests are undersized for small values of b in smaller samples and size distortions in these cases increase with correlation a. The LR test of Johansen and Nielsen (2012) heavily overrejects in all cases considered and size distortions increase with sample size T and correlation a, but decrease with b.

Table 1. Size simulation $k = 0$.

T	Test	b						
		0.40	**0.51**	**0.60**	**0.70**	**0.80**	**0.90**	**0.99**
50	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	2.0	2.3	2.4	2.3	3.1	2.7	3.0
	2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	1.8	2.2	2.5	2.5	2.9	2.6	2.8
	$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	2.4	3.4	4.0	4.4	5.1	4.8	5.1
	$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	2.4	3.7	4.1	4.6	5.2	4.9	5.3
	$\mathcal{LR}_T^3(1\|3)$, $d=1$	8.8	11.6	11.8	9.6	7.9	6.2	5.2
100	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	2.5	3.1	3.4	3.3	3.0	3.1	3.2
	2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	2.6	2.8	3.0	3.2	3.0	3.1	3.2
	$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	3.3	4.6	4.9	5.1	4.9	4.9	5.4
	$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	3.3	4.5	4.8	5.1	5.1	5.0	5.4
	$\mathcal{LR}_T^3(1\|3)$, $d=1$	9.3	10.6	11.3	9.5	7.7	6.2	5.4
200	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	3.1	3.1	3.2	2.9	3.0	3.1	2.8
	2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	3.0	3.0	3.2	2.8	3.0	3.1	2.7
	$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	4.4	4.5	4.8	4.8	5.2	5.2	4.6
	$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	4.4	4.5	4.9	4.8	5.2	5.3	4.8
	$\mathcal{LR}_T^3(1\|3)$, $d=1$	8.1	8.9	8.3	8.2	7.1	6.3	4.8
400	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	3.5	3.1	3.3	3.5	3.4	3.2	3.2
	2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	3.3	3.2	3.2	3.3	3.1	3.2	2.9
	$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	4.8	4.5	4.8	5.2	5.2	5.0	5.0
	$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	4.7	4.6	4.9	5.1	5.0	5.1	5.1
	$\mathcal{LR}_T^3(1\|3)$, $d=1$	6.5	7.6	7.6	7.0	6.3	5.8	5.1

Percentage of rejections by two step trace $\mathcal{LR}_T^2(0|2)$ and maximum eigenvalue test $\mathcal{LR}_T^2(0|1)$, $\mathcal{B} = [0.5, 1]$, Johansen's trace $\mathcal{LR}_T^3(1|3)$ and maximum eigenvalue $\mathcal{LR}_T^3(1|2)$ tests with $d = b = 1$ and trace test $\mathcal{LR}_T^3(1|3)$ of Johansen and Nielsen (2012) under the null hypothesis of cointegration rank $r = 1$ in a $p = 3$ dimensional system with $d = 1$, $k = 0$. Nominal size 5%.

Table 2. Size simulation $k = 1, a = 0.5$.

T	Test	b						
		0.40	**0.51**	**0.60**	**0.70**	**0.80**	**0.90**	**0.99**
50	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	7.2	7.2	8.2	8.7	9.9	10.3	10.7
	2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	7.4	7.1	8.4	8.5	9.8	10.6	10.6
	$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	1.5	1.5	2.1	2.6	3.4	4.2	5.0
	$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	1.7	1.7	2.6	3.1	3.8	4.6	5.3
	$\mathcal{LR}_T^3(1\|3)$, $d=1$	11.5	13.3	13.1	11.5	9.6	8.6	7.1
100	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	4.6	5.2	6.0	6.6	6.4	7.2	7.2
	2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	4.2	5.1	5.8	6.7	6.3	7.0	7.1
	$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	1.4	2.1	3.0	4.0	4.7	5.5	6.3
	$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	1.5	2.2	3.1	4.5	4.6	5.7	6.2
	$\mathcal{LR}_T^3(1\|3)$, $d=1$	17.7	20.6	22.6	18.8	14.9	11.5	8.2
200	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	4.3	5.0	5.7	5.5	4.9	4.8	4.6
	2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	4.2	4.8	5.2	5.3	4.8	4.8	4.5
	$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	2.1	3.2	4.4	5.4	5.4	5.4	5.1
	$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	2.2	3.3	4.4	5.1	5.5	5.5	5.3
	$\mathcal{LR}_T^3(1\|3)$, $d=1$	36.7	41.8	38.5	29.4	18.5	10.4	6.3
400	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	4.7	4.6	4.4	4.5	4.1	4.0	3.9
	2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	4.5	4.5	4.4	4.3	3.6	3.6	3.9
	$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	2.0	4.4	5.1	5.3	5.2	5.2	5.2
	$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	2.0	4.5	5.0	5.4	5.2	5.2	5.4
	$\mathcal{LR}_T^3(1\|3)$, $d=1$	57.9	67.3	55.8	36.8	17.0	8.3	5.7

Percentage of rejections by two step trace $\mathcal{LR}_T^2(0|2)$ and maximum eigenvalue test $\mathcal{LR}_T^2(0|1)$, $\mathcal{B} = [0.5, 1]$, Johansen's trace $\mathcal{LR}_T^3(1|3)$ and maximum eigenvalue $\mathcal{LR}_T^3(1|2)$ tests with $d = b = 1$ and trace test $\mathcal{LR}_T^3(1|3)$ of Johansen and Nielsen (2012) under the null hypothesis of cointegration rank $r = 1$ in a $p = 3$ dimensional system with $d = 1$, $k = 1$, $a = 0.5$. Nominal size 5%.

Table 3. Size simulation $k = 1, a = 0.8$.

T	Test	b						
		0.4	**0.51**	**0.60**	**0.70**	**0.80**	**0.90**	**0.99**
50	2s-$\mathcal{LR}_T^2(0\|1)$, $d = 1$	10.2	10.1	10.9	10.6	11.6	11.0	11.5
	2s-$\mathcal{LR}_T^2(0\|2)$, $d = 1$	10.8	10.1	10.9	10.9	11.7	11.4	12.0
	$\mathcal{LR}_T^3(1\|2)$, $d = b = 1$	3.1	2.9	2.7	3.0	3.3	3.4	4.1
	$\mathcal{LR}_T^3(1\|3)$, $d = b = 1$	3.7	3.3	3.2	3.7	3.8	4.1	4.9
	$\mathcal{LR}_T^3(1\|3)$, $d = 1$	33.6	32.9	31.5	27.0	22.6	20.6	18.4
100	2s-$\mathcal{LR}_T^2(0\|1)$, $d = 1$	5.1	5.5	6.2	6.6	6.9	7.9	7.8
	2s-$\mathcal{LR}_T^2(0\|2)$, $d = 1$	5.3	5.5	6.4	6.9	7.0	7.7	7.4
	$\mathcal{LR}_T^3(1\|2)$, $d = b = 1$	1.2	1.3	1.5	2.1	2.6	3.6	5.0
	$\mathcal{LR}_T^3(1\|3)$, $d = b = 1$	1.6	1.5	1.9	2.4	2.9	4.0	5.3
	$\mathcal{LR}_T^3(1\|3)$, $d = 1$	51.1	49.5	49.0	45.9	43.3	38.9	25.8
200	2s-$\mathcal{LR}_T^2(0\|1)$, $d = 1$	3.7	4.2	4.7	5.4	5.7	5.4	5.7
	2s-$\mathcal{LR}_T^2(0\|2)$, $d = 1$	3.6	4.2	4.5	5.2	5.7	5.4	5.7
	$\mathcal{LR}_T^3(1\|2)$, $d = b = 1$	0.8	1.1	1.8	2.6	4.2	5.3	5.8
	$\mathcal{LR}_T^3(1\|3)$, $d = b = 1$	1.0	1.4	1.9	2.9	4.7	5.4	6.0
	$\mathcal{LR}_T^3(1\|3)$, $d = 1$	83.2	83.1	83.9	82.9	79.3	53.4	29.8
400	2s-$\mathcal{LR}_T^2(0\|1)$, $d = 1$	3.7	4.0	4.6	5.0	4.4	4.5	4.3
	2s-$\mathcal{LR}_T^2(0\|2)$, $d = 1$	3.7	3.8	4.6	4.5	4.3	4.5	4.5
	$\mathcal{LR}_T^3(1\|2)$, $d = b = 1$	1.2	2.0	3.5	4.8	5.2	5.8	5.8
	$\mathcal{LR}_T^3(1\|3)$, $d = b = 1$	1.3	2.2	3.7	5.0	5.3	5.8	5.9
	$\mathcal{LR}_T^3(1\|3)$, $d = 1$	99.3	99.3	99.5	99.3	90.7	52.4	25.6

Percentage of rejections by two step trace $\mathcal{LR}_T^2(0|2)$ and maximum eigenvalue test $\mathcal{LR}_T^2(0|1)$, $\mathcal{B} = [0.5, 1]$, Johansen's trace $\mathcal{LR}_T^3(1|3)$ and maximum eigenvalue $\mathcal{LR}_T^3(1|2)$ tests with $d = b = 1$ and trace test $\mathcal{LR}_T^3(1|3)$ of Johansen and Nielsen (2012) under the null hypothesis of cointegration rank $r = 1$ in a $p = 3$ dimensional system with $d = 1$, $k = 1$, $a = 0.8$. Nominal size 5%.

Table 4. Power simulation $k = 0$.

T	b_1	Test	b						
			0.40	**0.51**	**0.60**	**0.70**	**0.80**	**0.90**	**0.99**
50	b	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	16.5	45.5	72.8	91.8	98.4	99.8	100
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	15.2	43.0	70.1	90.1	97.9	99.7	99.9
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	16.2	43.3	71.3	93.1	99.5	100	100
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	15.8	42.9	70.7	93.0	99.3	100	100
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	43.0	76.7	92.8	98.1	99.8	100	100
	0.20	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	3.7	6.4	7.5	8.4	9.4	9.6	9.3
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	3.7	5.8	7.2	7.8	8.4	8.9	9.0
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	4.4	7.7	9.2	10.4	11.5	12.1	11.9
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	4.7	7.3	9.4	10.1	11.1	12.2	12.0
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	17.3	22.6	23.8	20.1	17.5	15.2	12.6
	0.51	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	26.9	45.5	58.0	65.9	70.4	72.3	72.8
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	25.6	43.0	54.9	62.6	68.2	69.9	70.0
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	26.4	43.3	55.2	64.4	68.4	69.9	70.0
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	25.7	42.9	54.7	63.5	67.8	69.4	69.5
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	57.2	76.7	84.0	83.8	81.1	77.5	73.3
	0.90	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	41.2	68.2	86.2	95.7	99.2	99.8	99.9
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	39.1	66.4	84.1	95.0	98.7	99.7	99.9
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	42.9	69.3	87.1	97.3	99.8	100	100
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	42.7	69.0	86.9	97.3	99.7	100	100
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	68.0	89.8	97.1	99.3	99.9	100	100
100	b	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	72.7	97.7	98.8	100	100	100	100
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	20.9	96.9	98.8	100	100	100	100
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	50.2	88.1	98.8	100	100	100	100
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	49.3	88.3	98.9	100	100	100	100
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	91.0	99.7	100	100	100	100	100
	0.20	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	16.5	19.3	20.4	21.4	21.9	22.0	21.3
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	15.4	18.2	19.3	20.0	20.3	20.5	20.2
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	13.2	17.2	19.3	19.8	20.3	20.8	19.9
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	13.1	16.5	19.1	19.6	20.0	20.3	19.4
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	36.8	41.4	40.8	35.8	30.1	25.5	20.6
	0.51	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	84.5	97.7	99.0	99.5	99.6	99.5	100
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	82.0	96.9	98.8	99.2	99.4	99.4	100
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	67.2	88.1	94.0	95.4	95.6	95.7	100
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	66.9	88.3	94.2	95.4	95.9	95.7	100
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	95.7	99.7	99.9	99.8	99.5	99.0	100
	0.90	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	86.6	99.0	99.7	100	100	100	100
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	84.9	98.7	99.7	100	100	100	100
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	75.1	95.8	99.6	100	100	100	100
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	74.8	95.9	99.7	100	100	100	100
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	93.0	99.7	100	100	100	100	100

Percentage of rejections by two step trace $\mathcal{LR}_T^2(0|2)$ and maximum eigenvalue test $\mathcal{LR}_T^2(0|1)$, $\mathcal{B} = [0.5, 1]$, Johansen's trace $\mathcal{LR}_T^3(1|3)$ and maximum eigenvalue $\mathcal{LR}_T^3(1|2)$ tests with $d = b = 1$ and trace test $\mathcal{LR}_T^3(1|3)$ of Johansen and Nielsen (2012) under the alternative hypothesis of cointegration rank $r = 2$ in $p = 3$ dimensional system with $d = 1$, $k = 0$ and 2^{nd} cointegrating relationship with the memory $b_1 = b$, 0.20 ,0.51 or 0.9. Nominal size 5%.

Table 5. Power simulation $k = 1, a = 0.5$.

T	b_1	Test	b						
			0.40	**0.51**	**0.60**	**0.70**	**0.80**	**0.90**	**0.99**
50	b	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	9.0	9.4	13.4	19.4	29.2	41.9	52.8
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	9.1	9.8	13.7	20.2	29.3	42.2	52.2
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	2.8	2.3	4.0	7.9	15.1	28.2	44.5
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	3.1	2.7	4.5	8.9	16.5	29.2	45.0
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	24.2	15.4	19.7	23.3	27.1	35.7	45.2
	0.20	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	5.1	6.6	8.2	8.3	9.8	10.9	11.2
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	4.8	6.9	8.1	8.6	9.7	10.8	11.0
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	1.3	1.4	1.8	2.3	3.3	3.9	4.5
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	1.5	1.6	2.0	2.7	3.3	4.2	4.9
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	17.9	11.5	12.1	10.1	8.4	7.6	6.4
	0.51	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	5.1	9.4	11.3	12.8	15.3	17.4	17.6
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	4.8	9.8	11.6	13.4	15.5	17.7	17.1
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	1.3	2.3	2.9	4.0	5.5	7.3	9.1
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	1.5	2.7	3.3	4.9	5.9	8.4	9.7
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	17.9	15.4	16.5	15.2	12.7	12.3	10.8
	0.90	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	22.21	18.5	23.1	28.2	35.1	41.4	45.3
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	21.11	19.0	23.6	29.1	35.9	41.6	45.5
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	16.0	7.7	10.5	14.7	20.4	28.2	34.6
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	16.2	8.3	11.4	16.1	21.6	29.1	35.9
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	39.5	27.7	32.6	33.8	33.9	35.7	36.5
100	b	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	9.0	18.5	33.5	55.6	78.4	92.2	96.5
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	9.1	18.6	32.6	53.8	76.5	90.9	95.7
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	2.8	8.2	20.1	44.5	74.6	94.8	99.3
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	3.1	8.9	20.8	43.7	73.8	93.6	99.1
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	24.2	37.2	53.1	68.9	85.2	95.6	99.1
	0.20	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	5.1	6.0	7.6	8.6	9.7	10.2	9.6
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	5.0	6.1	7.3	8.4	9.6	10.1	9.3
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	3.4	4.1	4.8	5.7	6.3	6.7	6.2
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	3.6	4.0	4.6	5.7	6.3	6.8	6.1
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	17.8	31.4	21.9	19.7	15.3	12.9	9.4
	0.51	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	12.5	18.5	24.4	29.6	32.6	34.5	33.4
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	12.9	18.6	24.1	29.6	31.9	33.6	32.6
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	4.9	8.2	13.3	18.5	23.8	28.8	29.1
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	5.4	8.7	13.7	19.1	23.8	28.6	28.9
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	29.3	37.2	43.4	42.6	39.1	37.0	31.0
	0.90	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	22.2	37.2	53.9	70.9	85.2	92.2	93.9
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	22.1	36.8	52.3	69.3	83.8	90.9	93.0
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	16.0	28.4	44.6	66.7	84.6	94.8	97.2
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	16.2	29.3	44.6	65.5	83.8	93.6	96.6
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	39.5	56.8	72.2	83.9	91.4	95.6	96.6

Percentage of rejections by two step trace $\mathcal{LR}_T^2(0|2)$ and maximum eigenvalue test $\mathcal{LR}_T^2(0|1)$, $\mathcal{B} = [0.5, 1]$, Johansen's trace $\mathcal{LR}_T^3(1|3)$ and maximum eigenvalue $\mathcal{LR}_T^3(1|2)$ tests with $d = b = 1$ and trace test $\mathcal{LR}_T^3(1|3)$ of Johansen and Nielsen (2012) under the alternative hypothesis of cointegration rank $r = 2$ in $p = 3$ dimensional system with $d = 1$, $k = 1$, $a = 0.5$, and 2^{nd} cointegrating relationship with the memory $b_1 = b$, 0.20 , 0.51 or 0.9. Nominal size 5%.

Table 6. Power simulation $k=1, a=0.8$.

T	b_1	Test	b						
			0.40	**0.51**	**0.60**	**0.70**	**0.80**	**0.90**	**0.99**
50	b	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	3.2	6.2	6.2	7.3	8.7	12.4	14.5
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	3.3	6.6	6.4	7.8	9.3	13.2	15.4
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	3.0	1.6	1.4	1.7	2.2	3.5	5.3
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	3.0	1.8	1.8	2.2	2.8	4.9	7.0
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	32.9	19.4	17.5	12.3	12.5	13.1	13.4
	0.20	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	3.3	8.2	8.0	8.7	8.9	10.1	9.3
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	3.5	8.5	8.15	9.1	9.4	10.1	9.6
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	0.7	1.7	1.9	2.1	2.0	2.6	2.9
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	1.0	2.1	2.3	2.5	2.6	3.2	3.4
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	32.1	24.1	23.0	19.0	15.6	5.0	13.0
	0.51	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	6.5	6.2	6.4	7.2	7.5	7.8	8.3
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	6.6	6.6	6.9	7.3	7.8	8.4	8.7
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	1.5	1.6	1.4	1.6	1.8	1.9	2.2
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	1.9	1.8	1.8	2.0	2.0	2.3	2.7
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	19.5	19.4	17.8	14.5	11.1	10.9	9.5
	0.90	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	7.3	8.0	7.9	8.8	9.9	12.0	12.7
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	7.6	8.3	8.6	9.1	10.3	12.7	13.7
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	2.0	2.1	2.1	2.2	2.7	3.5	4.1
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	2.4	2.5	2.6	2.8	3.7	4.9	5.4
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	21.8	21.9	20.0	16.8	14.0	13.1	12.1
100	b	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	3.2	3.3	5.2	7.4	14.1	25.6	42.9
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	3.2	3.6	5.5	8.1	14.9	26.2	40.8
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	0.7	3.2	4.3	5.6	7.5	9.4	12.0
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	1.0	3.2	4.4	5.9	7.5	9.4	11.9
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	32.9	32.6	33.6	31.9	32.9	34.0	38.8
	0.20	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	3.8	3.7	4.5	4.9	5.4	5.7	6.2
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	3.9	3.6	4.7	4.8	5.5	5.4	6.1
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	0.9	1.0	1.1	1.5	1.8	5.0	5.0
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	1.0	1.1	1.4	2.0	2.1	4.7	4.9
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	39.2	38.4	38.4	34.8	31.3	30.6	20.4
	0.51	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	3.3	3.3	4.4	5.1	5.9	7.2	8.1
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	3.5	3.6	4.5	5.4	6.5	7.5	8.1
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	0.7	3,2	1.0	1.4	1.9	3.1	4.2
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	1.0	3.2	1.4	1.9	2.6	3.7	5.0
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	32.1	32.6	32.7	29.1	25.8	25.5	17.9
	0.90	2s-$\mathcal{LR}_T^2(0\|1)$, $d=1$	6.7	8.2	10.1	13.8	19.4	25.6	31.7
		2s-$\mathcal{LR}_T^2(0\|2)$, $d=1$	7.0	8.3	10.7	14.3	19.6	26.1	31.6
		$\mathcal{LR}_T^3(1\|2)$, $d=b=1$	2.6	3.1	3.9	5.6	9.3	15.4	21.2
		$\mathcal{LR}_T^3(1\|3)$, $d=b=1$	3.2	3.7	5.2	7.4	11.2	17.5	22.8
		$\mathcal{LR}_T^3(1\|3)$, $d=1$	41.6	42.6	41.9	39.3	36.9	34.0	31.1

Percentage of rejections by two step trace $LR_T^2(0|2)$ and maximum eigenvalue test $LR_T^2(0|1)$, $B=[0.5,1]$, Johansen's trace $LR_T^3(1|3)$ and maximum eigenvalue $LR_T^3(1|2)$ tests with $d=b=1$ and trace test $LR_T^3(1|3)$ of Johansen and Nielsen (2012) under the alternative hypothesis of cointegration rank $r=2$ in $p=3$ dimensional system with $d=1$, $k=1$, $a=0.8$, and 2^{nd} cointegrating relationship with the memory $b_1=b$, 0.20 , 0.51 or 0.9. Nominal size 5%.

The simulated power is reported in Tables 4-6 for $T = 50$ and 100. When $k = 0$, see Table 4, all procedures have very good power for all sample sizes T and all true values of b, b_1, except $b_1 = 0.20$, which is very difficult to detect when $T = 50$. The power of all tests is increasing with sample size T, b and b_1, two step method doing marginally better for small b and/or b_1, while Johansen tests do better when $T = 50$ and $b_1 \geq 0.9$, since this case is closer to the unit root model assumed by these tests. When $k = 1$ all procedures are much less powerful than for $k = 0$, especially for the larger value of the autoregressive coefficient a and small b, see Tables 5 and 6. However still power increases with sample size T, b and b_1 for all methods. Two step procedures are noticeably more powerful than Johansen tests except of the cases close to $b = b_1 = 1$ in sample $T = 100$. The LR test of Johansen and Nielsen (2012) has largest power among all in many parameter combinations, but it is not relevant as this test does not keep the size in this experiment. To sum up, two step rank tests have a similar behavior to the one-step LR test when $d = b \approx 1$, however they seem to be more powerful when b and/or b_1 are small, being able to exploit the differences between b and b_1 or (b, b_1) and d, which are fixed in Johansen (1998) and Johansen and Nielsen (2012) methodologies.

6 Analysis of the term structure of the interest rates

To illustrate the empirical relevance of the described methodology we reconsider the analysis of the term structure of the interest rates by Iacone (2009). There has been a lot of interest in this issue in the current literature, see for example Chen and Hurvich (2003) and Nielsen (2010).

As argued in Iacone (2009), a good model of the term structure of the interest rates is needed to measure the effects of the monetary policy and to price financial assets. It is an important tool for policy evaluation since the Federal Reserve operates in just one market, the one with contracts with very short maturity. Therefore, it is necessary to model the conduction of the monetary policy impulses to the rates of contracts with longer maturities. Modeling the interactions across rates is also important for the economic agents to forecast the effects of future monetary policy decisions on the price of financial assets. Soderlind and Svensson (1997) have discussed a practical example of how to extract the market's expectations on future policy rates from a given term structure, and how to use them to price financial instruments.

Cointegration has an appealing feature in the analysis of the term structure, because it makes possible to distinguish the high persistence of shocks to interest rates from the much lower persistence of shocks to the spreads. Standard cointegration in the context of modeling a vector of US dollar interest rates has been considered by Hall, Anderson and Granger (1992), Engsted and Tangaard (1994), Dominguez and Novales (2000).

However it has been argued that the unit root model for the interest rates is often incompatible with monetary and finance theories, because it may imply a unit root model for the expected inflation rate as well. This is the case, for example, if the real interest rate is constant in the long run, or if the central bank sets the interest rate using a linear reaction function like the ones described by Taylor (1993) or by Svensson (1997). Such a strong persistence is hardly acceptable, because it implies that the central bank does not stabilize inflation.

We can allow for fractional cointegration instead. It permits to combine high persistence with mean reversion in the long run, and it maintains the possibility of the presence of a common stochastic terms in multivariate processes. Fractional integration may be motivated as the result of occasional breaks in an otherwise weakly autocorrelated process. This interpretation seems particularly appealing when modeling the interest rates because changes to the discount rate are infrequent. Granger and Hyoung (2004) have shown that fractional integration and occasional breaks may in practice be indistinguishable and, following also a comment by Diebold and Inoue (2001), adopting fractional integration in a model may result in good forecasts.

We analyze the behavior of the US dollar interest rates with maturities of 1, 3 and 6 months (the London InterBank Offered Rate LIBOR) over the period 01/1963-04/2006. The data come from DataStream with identification codes being respectively USI60LDC, USI60LDD, USI60LDE. LIBOR is not affected by any regulation imposed by the central bank, and thus it is a typical measure of the cost of funds in US dollars. For this data set Iacone (2009) has found evidence that the three considered series share the same order of integration with estimated $\hat{d} = 0.88$. The test of Robinson and Yajima (2002) and local Whittle procedure of Robinson (1995) have been used to obtain this result. Iacone (2009) has also concluded the fractional cointegration with rank $r = 2$ in this system using procedures in Phillips and Ouliaris (1988) and Robinson and Yajima (2002).

However the integration order of the cointegrating residuals of two relations found by Iacone (2009) differ significantly, and the transmission of impulses is slower the longer distance (in maturity) from the market where the Federal Reserve is directly present, so a model that allows different b's would be appropriate for this example. Łasak (2008) has analyzed three bivariate systems and has not imposed the assumption that both cointegration relationships share the same memory. The methodology developed in this paper enables us to test the rank directly in the 3-variate system (1) without imposing such assumption, as we pursue.

We consider the basic version of the model presented in Section 3, as it seems to be a right choice looking at PACF of the processes. We have tested the existence of the breaks in levels of considered series using the test of Sibbertsen and Kruse (2009) and it has indicated no breaks in the series. All the tests considered in Section 5 have been computed and all confirm that this system is cointegrated with rank 2. The values of the test statistics when testing rank $r = 1$ are presented in Table 7.

Table 7. Rank tests statistics under $H_0 : r = 1,\ d = \hat{d} = 0.88$

LR test	2-step	Johansen	J-N
max lambda	19.04	53.8	-
trace	19.04	54.5	30.7

In Table 8 we provide the 5% critical values for two step tests when $d = 0.88$ and $\mathcal{B} = [0.5, 0.88]$, compared to those when $d = 1$ and $\mathcal{B} = [0.5, 1]$. The critical values for $d < 1$ are smaller than those for $d = 1$, and in general, using the latter for situations when $d < 1$ would lead to a conservative inference. In any case, the tests statistics in Table 7 are significant at the 5% level even using the conservative critical values for $d = 1$.

Table 8. 5% Critical values of rank tests under $H_0 : r = 1$, $p = 3$.

LR test	2-step ($\mathcal{B}$= [0.5,1])	2-step ($\mathcal{B}$= [0.5,0.88])	Johansen	J-N
max lambda	11.72	11.02	11.23	-
trace	12.84	12.13	12.32	10.95

We also estimate the cointegration vectors on the basis of all considered models, including the VECM with $d = b = 1$ and the FVECM (1) with $d = \hat{d} = 0.88$ imposed, which is justified by Corollary 2. The first cointegration relationship is common to all procedures, but the second one can be different. When we focus on the two-step procedure proposed in Section 3, the estimate of the second cointegrating relationship β_1 is found according to the formula $\hat{\beta}_1 = \beta'_{\perp}\beta_1^*$, where β_1^* comes directly from solving the eigenvalue problem (2) constructed on the basis of the transformed model (16). It turns out that the outcomes of all the procedures imply the same cointegrating space spanned by

$$\hat{\beta}^{norm} = \begin{bmatrix} 1 & 1 \\ -0.98 & 0 \\ 0 & -0.96 \end{bmatrix}.$$

The cointegrating parameters are very close to -1, so the spreads can be computed as $s_t^{(j)} = i_t^{(j)} - i_t^{(1)}$, $j = 3, 6$. Iacone (2009) has estimated the orders of integration of these spreads using Local Whittle estimator of Robinson (1995) to be $s_t^{(j)} \sim I(d_j - b_j)$, $s_t^{(3)} \sim I(0.34)$ and $s_t^{(6)} \sim I(0.47)$ and rejected the hypothesis that these orders are the same. Therefore the rank estimation methodology developed in this paper is suitable for this example, as it takes into account the possibility that the persistence of the cointegration relationships differ. However when we estimated the FVECM using ML our results do not confirm that the spreads are persistent.

Table 9. Estimates of b and b_1, $d = \hat{d} = 0.88$.

	1st step ($\hat{b}$)	2nd step ($\hat{b}_1$)	J-N
$\hat{b}$	0.81	0.88	0.83

Looking at the estimates of the order of cointegration b in Table 9 we might conclude that the spreads seem to behave as $I(0)$ processes, so the evidence supporting the Expectation Hypothesis can be found in the multivariate case if the analysis does not restrict all the cointegration relationships to share the same memory.

7 Conclusions

In this paper we have proposed a new procedure, based on sequential two-step LR tests, to establish the cointegration rank in a fractional system. The main novelty is that it allows the cointegrating relationships under the alternative to have different memory compared to the null ones. It only needs a small modification of the model estimated in the second step. The asymptotic distributions of the test statistics are the same as for the no-cointegration testing, so the set of simulations required to approximate the critical values is reduced, which can be seen as an advantage for empirical work. We have investigated the performance of our procedure in finite samples and have compared it with the LR trace test of Johansen and Nielsen (2012) and with Johansen's LR trace and maximum eigenvalue tests. We have found that our tests control size and have an advantage in terms of power to detect extra cointegrating relationships in situations when the memories of the cointegration relations differ or are relatively small.

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Appendix

Proof of Theorem 1. We demonstrate first that replacing $\beta_\perp$ by $\hat{\beta}_\perp$ makes no difference asymptotically in the two step LR test statistics. LR tests statistics depend on properly normalized sample moments of dependent and independent variables in the regression model (16), cf. (2). The result follows from Theorem 1 in Łasak (2010), after controlling for the projection on $\left(\Delta_+^b - 1\right) u_{1t}$ as in Lemma 10.3 in Johansen (1995), using the representation (14) for the dependent variable $V_{0t} = \beta'_\perp \Delta^d X_t$.

Set $V_{1t}(b_1) = \left(1 - \Delta_+^{-b_1}\right) V_{0t}$, recalling the definition of $\hat{V}_{1t}(b_1)$ and using the true $\beta_\perp$. First, we want to show that

$$T^{-b_1} \sum_{t=1}^{T} \hat{V}_{1t}(b_1) \hat{V}'_{0t} - T^{-b_1} \sum_{t=1}^{T} V_{1t}(b_1) V'_{0t} \to_p 0$$

uniformly for $b_1 \in \mathcal{B}$ if $\hat{\beta}_\perp - \beta_\perp = O_p\left(T^{-b}\right)$. The difference on the left hand side is

$$T^{-b_1} \sum_{t=1}^{T} \left\{\hat{V}_{1t}(b_1) - V_{1t}(b_1)\right\} V'_{0t} + T^{-b_1} \sum_{t=1}^{T} \hat{V}_{1t}(b_1) \left(\hat{V}_{0t} - V_{0t}\right)'. \tag{23}$$

The first term in (23) is equal to

$$\left(\hat{\beta}'_\perp - \beta'_\perp\right) T^{-b_1} \sum_{t=1}^{T} \left(1 - \Delta_+^{-b_1}\right) \Delta^d X_t V'_{0t} = o_p(1),$$

uniformly in $b_1 \in \mathcal{B}$ because $\hat{\beta}_\perp - \beta_\perp = O_p\left(T^{-b}\right)$, $b > 0.5$, and $T^{-b_1} \sum_{t=1}^{T} \left(1 - \Delta_+^{-b_1}\right) \Delta^d X_t V'_{0t} = O_p\left(T^{1/2-\epsilon}\right)$ uniformly in b_1, $b_1 > 0.5$, for some $\epsilon > 0$ from (104) in Lemma A.9 in Johansen and Nielsen (2012).

The second term on the right hand side of (23) is

$$T^{-b_1} \sum_{t=1}^{T} \hat{V}_{1t}(b_1) \Delta^d X'_t \left(\hat{\beta}_\perp - \beta_\perp\right) = O_p\left(T^{-b}\right) T^{-b_1} \sum_{t=1}^{T} \hat{V}_{1t}(b_1) \Delta^d X_t,$$

and this is $O_p\left(T^{-b}\right) O_p\left(T^{1/2-\epsilon}\right) = o_p(1)$, uniformly in b_1 with $b > 0.5$, $\epsilon > 0$, using again Lemma A.9 in Johansen and Nielsen (2012).

Using the same ideas it can be shown that

$$T^{-2b_1}\sum_{t=1}^{T}\hat{V}_{1t}(b_1)\hat{V}_{1t}'(b_1)-T^{-2b_1}\sum_{t=1}^{T}V_{1t}(b_1)V_{1t}'(b_1)\rightarrow_p 0$$

uniformly for $b_1\in\mathcal{B}$ and

$$T^{-1}\sum_{t=1}^{T}\hat{V}_{0t}\hat{V}_{0t}'-T^{-1}\sum_{t=1}^{T}V_{0t}V_{0t}'\rightarrow_p 0,$$

exploiting (103) and (102), respectively, in Lemma A.9 in Johansen and Nielsen (2012), so that the estimation of $\beta_\perp$ in the first step has no impact on the asymptotic distribution of the test statistics.

We next show that replacing $\left(\Delta_+^b-1\right)u_{1t}$ by $\left(\Delta_+^{\hat{b}}-1\right)\hat{u}_{1t}=\left(1-\Delta_+^{-\hat{b}}\right)\hat{\beta}'\Delta^d X_t$ in (16) is also negligible asymptotically under (12). For that, it is enough to consider the differences

$$T^{-b_1}\sum_{t=1}^{T}V_{1t}(b_1)\left(\Delta_+^b-1\right)u_{1t}'-T^{-b_1}\sum_{t=1}^{T}V_{1t}(b_1)\left(\Delta_+^{\hat{b}}-1\right)\hat{u}_{1t}' \tag{24}$$

$$T^{-b_1}\sum_{t=1}^{T}u_t\left(\Delta_+^b-1\right)u_{1t}'-T^{-b_1}\sum_{t=1}^{T}u_t\left(\Delta_+^{\hat{b}}-1\right)\hat{u}_{1t}', \tag{25}$$

since other terms appearing in the projections of $V_{1t}(b_1)$ and V_{0t} on $\left(\Delta_+^{\hat{b}}-1\right)\hat{u}_{1t}$ could be dealt with in the same way. We can decompose (24) in

$$T^{-b_1}\sum_{t=1}^{T}V_{1t}(b_1)\left(\Delta_+^b-\Delta_+^{\hat{b}}\right)\hat{u}_{1t}'+T^{-b_1}\sum_{t=1}^{T}V_{1t}(b_1)\left(\Delta_+^{\hat{b}}-1\right)\Delta^d X_t'\left\{\beta-\hat{\beta}\right\}. \tag{26}$$

The first term in (26) can be shown to be $o_p(1)$ uniformly in b_1 as in Robinson and Hualde (2003, Proposition 9), expanding $\left(\Delta_+^b-\Delta_+^{\hat{b}}\right)u_{1t}=\left(1-\Delta_+^{\hat{b}-b}\right)\Delta_+^b u_{1t}$ around $b-\hat{b}=0$, with $b-\hat{b}=O_p\left(T^{-1/2}\right)$ and noting that the terms in the expansion behave as the derivatives of $\Delta_+^b u_{1t}$ with respect to b, cf. (104) in Lemma A.9 in Johansen and Nielsen (2012), whose sample moments are $O_p\left(T^{1/2-\epsilon}\right)$ uniformly in b_1, $\epsilon>0$. The second term in (26) is $o_p(1)$ using a similar argument for $\left(\Delta_+^{\hat{b}}-1\right)\Delta^d X_t$, being approximately an $I(-b)$ asymptotically stationary process, and the superconsistency of $\hat{\beta}$. Finally, the analysis of (25) being $o_p(1)$ is simpler because it does not depend on b_1 and u_t is i.i.d.

Then to show the validity of the correction introduced in regression (16), it is only necessary to observe that the vector u_t is just a rotation of the vector ε_t, so all previous approximations and bounds can be used similarly. ■

Proof of Corollary 2. We have to additionally show that terms like

$$T^{-b_1} \sum_{t=1}^{T} V_{1t}\left(b_1, \hat{d}\right) V_{0t}'\left(\hat{d}\right) - T^{-b_1} \sum_{t=1}^{T} V_{1t}\left(b_1, d\right) V_{0t}'\left(d\right)$$

are $o_p(1)$ uniformly for $b_1 \in \mathcal{B}$ if $\hat{d} - d = O_p\left(T^{-1/2}\right)$. This follows from a similar analysis as that of the first term in (26), writing this difference as

$$T^{-b_1} \sum_{t=1}^{T} V_{1t}\left(b_1, \hat{d}\right) \left\{V_{0t}'\left(\hat{d}\right) - V_{0t}'\left(d\right)\right\} + T^{-b_1} \sum_{t=1}^{T} \left\{V_{1t}\left(b_1, \hat{d}\right) - V_{1t}\left(b_1, d\right)\right\} V_{0t}'\left(d\right)$$

and using a Taylor expansion of $1-\Delta_+^{\hat{d}-d}$ around $\hat{d}-d = 0$ in $V_{0t}'\left(\hat{d}\right)-V_{0t}'\left(d\right) = \left(1 - \Delta_+^{\hat{d}-d}\right)\Delta^d \beta_\perp' X_t$ and $V_{1t}\left(b_1, \hat{d}\right)-V_{1t}\left(b_1, d\right) = (1-\Delta_+^{-b_1})\left(1 - \Delta_+^{\hat{d}-d}\right)\Delta^d \beta_\perp' X_t$, and then using uniform bounds for the corresponding sample moments on (derivatives of) fractionally integrated processes. ■

Proof of Theorem 3. The proof follows the lines of the proof of Theorem 1, since the additional lags $\Delta^d X_{t-j}$, $j = 1, \ldots, k$ in regression (20) pose no additional problem compared to the projection of $\hat{V}_{0t}$ and $\hat{V}_{1t}(b_1)$ on $\left(\Delta_+^{\hat{b}} - 1\right)\hat{u}_{1t}$, because the former are observed and $I(0)$. ■

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